On the domain of variability of coefficients of power series which do not take given values

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Introduction.

If we are given an analytic function y of the complex variable z which takes the value $y = A_0$ for $z = 0$ and is regular in the neighbourhood of this point, but is subject to certain restrictions in the interior of the circle $|z| < \rho$, then there stands the question of whether or not thereby there also arise restrictions on the coefficients of the power series *p.95*

$$y = A_0 + \sum_{i=1}^{\infty} A_k z^k,$$

that represents the function, which can be determined.

A special case of this type of question occurs in the well-known generalisation to all transcendental functions which E. Landau has given for Picard's Theorem.*

Indeed, this theorem can be described in the following way: if the function y takes the value $y = A_0$ for $z = 0$, is regular in the interior of the unit disc and leaves out the values zero and one, and if we refer to the real and imaginary part of the coefficient A_1 as coordinates of a point in the plane, then this point must lie in the interior of a circle, whose radius can be given. *p.96*

*E. Landau, Sitzungsb. d. Berl. Akad. (1904), XXXVIII (pag. 1118), see also A. Hurwitz, Züricher Vierteljahrsschr. XLIX (pag. 242); F. Schottky, Berl. Akad. (1904) XLII (pag. 1244); C. Carathéodory, C. R. Bd. 141 (1905), p. 1213; P. Boutroux, Bull. Soc. Math. Bd. 34 (1906), p. 30.
During the printing of this work Mr Landau has published a detailed representation of the question in our consideration. In particular, in §15 of this work a problem analogous to the one we consider here is resolved for a recurring algebraic procedure.

In a similar way, in the following I shall refer to the real and imaginary parts of the n coefficients

$$A_1,\ A_2,\ \ldots,\ A_n$$

as coordinates of a point in $2n$-dimensional space and call this point the *n-th geometrical representative* of the power series. Then it will be shown that, if the function y satisfies prescribed conditions, similar to in the Landau–Picard Theorem, then this point must lie in the interior or on the surface of a body K_{2n}, which one can determine completely and explicitly.

We shall explore the special cases where the function y is regular in the interior of the unit disc and has a positive real part, while it takes the value $y = 1$ for the point $z = 0$.

If we represent by $\mathfrak{C}_n$ the curve which is traced by the points with the $2n$ coordinates

$$\begin{array}{c} 2\cos\theta,\quad 2\cos 2\theta,\ \ldots,\quad 2\cos n\theta, \\ -2\sin\theta, -2\sin 2\theta,\ \ldots, -2\sin n\theta \end{array}$$

as θ varies from 0 to 2π, then the body K_n coincides with the smallest convex body $\mathfrak{K}_n$ which contains $\mathfrak{C}_n$.

Conversely, however, every point of $\mathfrak{K}_n$ is also the n-th geometrical representative of at least one function $y(z)$ which satisfies the prescribed conditions; this function is uniquely determined, rational, and possesses at most n zeroes for the points on the surface.*

The general question will be resolved from this special one with the help of a suitable conformal mapping; this mapping corresponds to a birational transformation of the n-th geometrical representative in $2n$-dimensional space, and the body K_n sought is nothing other than the transformation of the convex body $\mathfrak{K}_n$.

§1.
Definition of the body $\mathfrak{K}_n$.

We first consider functions of a complex argument z which are regular in the interior *and on the edge* of the unit disc

$$|z| \leqslant 1,$$

have a positive real part for these values of z, and for $z = 0$ take the value one. A function which satisfies all these conditions, may be represented in the
p.97 neighbourhood of $z = 0$ by a power series

*This result is, in other words, a generalisation and specialisation of the well-known Hadamard–Borel inequalities.

$$y = 1 + \sum_{k=1}^{\infty} (c_k + i\overline{c}_k) z^k \tag{1}$$

whose radius of convergence is greater than one; by $f(\theta)$ we denote the real part of (1) on the unit disc

$$z = e^{i\theta};$$

this quantity, which by our assumptions must satisfy the relation

$$f(\theta) \geqslant 0, \tag{2}$$

may be expanded as a convergent Fourier series

$$f(\theta) = 1 + \sum_{k=1}^{\infty} (c_k \cos k\theta - \overline{c}_k \sin k\theta); \tag{3}$$

therefore the well-known equations

$$\frac{1}{\pi} \int_0^{2\pi} f(\theta)\, d\theta = 2, \tag{4}$$

$$\begin{cases} \dfrac{1}{\pi} \displaystyle\int_0^{2\pi} f(\theta) \cos k\theta\, d\theta = c_k, \\ \dfrac{1}{\pi} \displaystyle\int_0^{2\pi} f(\theta) \sin k\theta\, d\theta = -\overline{c}_k, \end{cases} \tag{5}$$

hold.

From the equations (5), and indeed for every k, the relations

$$|c_k| \leqslant 2, \quad |\overline{c}_k| \leqslant 2$$

follow by applying the Intermediate Value Theorem with the help of (4) and (2). The n-th geometrical representative of a function which satisfies our requirements is therefore always bounded.

The degree n polynomial

$$y = 1 + \sum_{k=1}^{n} (c_k + i\overline{c}_k) z^k \tag{6}$$

has the point with coordinates

$$\begin{cases} c_1,\ c_2,\ \ldots,\ c_n, \\ \overline{c}_1,\ \overline{c}_2,\ \ldots,\ \overline{c}_n \end{cases} \tag{7}$$

as its n-th geometrical representative, and because the real part of the function (6) has the value

$$f(\theta) = 1 + \sum_{k=1}^{n} (c_k \cos k\theta - \bar{c}_k \sin k\theta)$$

on the unit disc, we see that this real part cannot be negative on the whole *p.98*
surface of the circle if for every k we have

$$|c_k| < \frac{1}{2n}, \quad |\bar{c}_k| < \frac{1}{2n}.$$

Because in this case the function (6) satisfies all our conditions, we see that certain neighbourhoods of the origin are consist of geometrical representatives of our functions; one such neighbourhood is e.g. the ball of radius $\frac{1}{2n}$

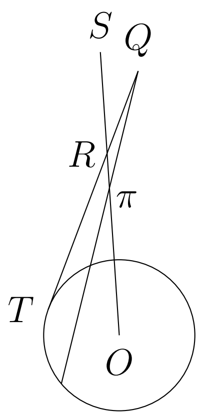
$$\sum_{k=1}^{n} (c_k^2 + \bar{c}_k^2) \leqslant \frac{1}{4n^2}. \tag{8}$$

If y_1 and y_2 are two functions which satisfy all of our requirements, then the same holds for every function y from the collection

$$\begin{cases} y = ty_1 + (1-t)y_2, \\ \qquad 0 \leqslant t \leqslant 1; \end{cases}$$

the representative P of y traces out the line segment which connects the representatives P_1 and P_2 of y_1 and y_2, as t varies from 0 to 1; this segment therefore consists exclusively of geometrical representatives of our functions.

Figure 1



It follows above all from this remark that for every radius vector of $2n$-dimensional space which emanates from the origin, a unique boundary point π exists, which separates those points of this line which can be representatives of one of our functions, from the others which do not have this property. Now let S be a point on the ray which connects the origin O with the boundary point π and let π lie between O and S (Fig. 1). Every point Q which differs from S by less than

$$\frac{1}{2n} \frac{\pi S}{O\pi},$$

cannot possibly be a representative of one of our functions. Because the line $Q\pi$ cuts through the circle of radius $\frac{1}{2n}$, which determines the $2n$-dimensional ball (8) in the plane OQS (Fig. 1); one can therefore also place a tangent QT to this circle from Q, which meets the segment in an interior point R. If now Q were a representative of a function

$y(z)$ which satisfies our requirements, then the same would have to hold for every point of the segment QT, and consequently also for R, and π would not be a p.99 boundary point of the representatives on the segment OS.

We denote by $\mathfrak{K}_n$ the set of points that one obtains if one adds the boundary points π to the representatives of functions satisfying our requirements; it follows from the foregoing that every point S which is not contained in $\mathfrak{K}_n$ is separated from this set by a finitely determined distance; because now the collection of points $\mathfrak{K}_n$ contains no isolated points, we see that it is perfect.

If P_1 and P_2 are two arbitrary points of $\mathfrak{K}_n$, then every point of the line segment P_1P_2 also belongs to this set. Indeed, if a point S of this segment were to lie outside $\mathfrak{K}_n$, then there would have to exist a parallel segment $P_1'P_2'$ to P_1P_2 in the triangle OP_1P_2 which contained at least one point S' that did not belong to $\mathfrak{K}_n$, and this is impossible, because P_1' and P_2' are certainly representatives of functions $y(z)$ which satisfy all of our earlier requirements.

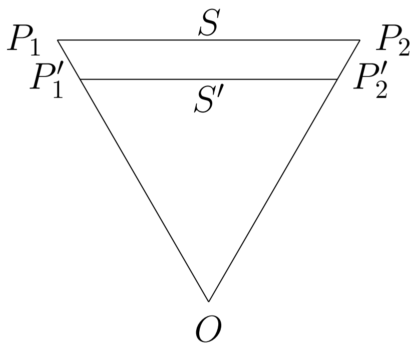


Figure 2

The collection of points $\mathfrak{K}_n$ therefore satisfies the definition of a $2n$-dimensional convex body that Minkowski established:* it does not entirely lie in a $(2n-1)$-dimensional hyperplane, it is perfect, and it contains every linear segment whose endpoints it contains.

The surface $\mathfrak{D}_n$ of the convex body $\mathfrak{K}_n$ is formed of the totality of the boundary points π, of which there is one lying on every radius vector from the origin; because one can show, and in fact in a very analogous way to earlier, for the exterior points S, that, if P is an interior point of the segment $O\pi$, a certain neighbourhood of P consists exclusively of points of $\mathfrak{K}_n$ and consequently P is also an interior point of $\mathfrak{K}_n$; in this way $\mathfrak{D}_n$ completely determines the body $\mathfrak{K}_n$.

§2.
Supporting hyperplanes.

A closed set of points in n-dimensional space possesses the familiar *supporting hyperplanes*,** i.e. such $(n-1)$-dimensional planes which contain at least one point of the set and which divide n-dimensional space lying outside the hyperplane into two parts, one of which does not contain a single point of the set.

From the work of Mr Minkowski it follows that, for a convex body $\mathfrak{K}$, there is

*Geometrie der Zahlen (Leipzig 1896) p.200.
**This expression was introduced by Minkowski *loc. cit.* p.13.

at least one supporting hyperplane through every point of the surface $\mathfrak{O}$. I owe
p.100 the idea of the following simple proof of this fact to his friendly message.

Let π be an arbitrary point of the surface $\mathfrak{O}$ of $\mathfrak{K}$ with coordinates

$$\gamma_1,\ \gamma_2,\ \ldots,\ \gamma_n;$$

furthermore, let S be an arbitrary point which lies outside the body $\mathfrak{K}$, and ϵ be the distance between π and S. The distance of S to an arbitrary point of $\mathfrak{K}$ has a non-zero minimum δ that is attained for a point π' of the body, and we always have

$$\delta \leqslant \epsilon.$$

The $(n-1)$-dimensional plane which contains π' and intersects the segment $\pi' S$ perpendicularly is a supporting hyperplane of the body $\mathfrak{K}$, because if a point T of $\mathfrak{K}$ were to lie outside this hyperplane and on the same side as S, then due to the convexity of $\mathfrak{K}$, the whole segment $\pi' T$ consists exclusively of points of this body, and the minimum of the distance between S and $\mathfrak{K}$ would not be attained by the point π', which follows immediately from considering the triangle $S\pi' T$. Because now, consequently, π comes to lie on the opposite side as S, if it is not a point of this hyperplane, then the segment πS certainly contains a point of the supporting hyperplane constructed and this is at a distance of less than ϵ from π. We now consider a series of points

$$S_1,\ S_2,\ S_3,\ \ldots, \tag{9}$$

which all lie outside the body $\mathfrak{K}$ and converge to π and let

$$u_1^{(k)} c_1 + u_2^{(k)} c_2 + \cdots + u_n^{(k)} c_n - d = 0 \tag{10}$$

be the equation of the supporting hyperplane constructed using the point S_k in the way depicted above. If the equation (10) is written in normal form, i.e. if the coefficients $u_i^{(k)}$ satisfy the condition

$$\sum_{i=1}^{n} (u_i^{(k)})^2 = 1, \tag{11}$$

then the left side of (10) represents the distance of the point with coordinates $c_1, c_2, \ldots$ from the said hyperplane. Therefore if we denote by ϵ_k the distance between S_k and the boundary point π, whose coordinates were γ_i, then we have by the foregoing

$$u_1^{(k)} \gamma_1 + u_2^{(k)} \gamma_2 + \cdots + u_n^{(k)} \gamma_n - d < \epsilon_k;$$

and so the equation (10) may be written

$$u_1^{(k)} (\gamma_1 - c_1) + u_2^{(k)} (\gamma_2 - c_2) + \cdots + u_n^{(k)} (\gamma_n - c_n) - h_k = 0 \tag{12}$$

in which

$$h_k < \epsilon_k$$

and consequently the equation p.101

(13)
$$\lim_{k\to\infty} h_k = 0$$

holds. Because, due to (11), all quantities $u_i^{(k)}$ are now bounded, we may select a new series from the points $S_1, S_2, \ldots$, for which the quantity $\lim_{k\to\infty} u_i^{(k)}$ exists, so that one can write the equation

$$\lim_{k\to\infty} u_i^{(k)} = u_i$$

for every

$$i = 1, 2, \ldots, n.$$

I claim that the hyperplane

(14)
$$u_1(\gamma_1 - c_1) + u_2(\gamma_2 - c_2) + \cdots + u_n(\gamma_n - c_n) = 0,$$

which contains the point π, is a supporting hyperplane. Indeed, in the opposite case there would exist two points P and P' of $\mathfrak{K}$ whose coordinates would give opposite signs when inserted into the left side of (14). However, the same would then apply for an appropriate choice of k in (12), which would not be reconcilable with the fact that for every k the equation (12) represents a supporting hyperplane, and hereby the theorem of Minkowski is proven.

§3.
Determination of the surface.

We now return to the convex body $\mathfrak{K}_n$ in $2n$-dimensional space, which we considered in the first section. Every *interior* point of $\mathfrak{K}_n$ is the n-th geometrical representative of a function $y(z)$, whose radius of convergence is larger than one, whose real part is non-negative in the unit disc, and which takes the value one for $z = 0$. If now, conversely,

(15)
$$y = 1 + \sum_{k=1}^{\infty} (c_k + i\bar{c}_k) z^k$$

is a function whose radius of convergence is greater than *or equal* to one and whose real part is non-negative in the unit disc, then for all

$$r < 1$$

the point with coordinates

$$\begin{cases} c_1 r,\ c_2 r^2,\ \dots,\ c_n r^n, \\ \overline{c}_1 r,\ \overline{c}_2 r^2,\ \dots,\ \overline{c}_n r^n \end{cases}$$

lies in the interior of the body $\mathfrak{K}_n$ and consequently the point with coordinates

$$c_1,\ c_2,\ \dots,\ c_n; \quad \overline{c}_1,\ \overline{c}_2,\ \dots,\ \overline{c}_n$$

p.102 in the interior or on the surface of this perfect set of points; the n-th geometrical representative of (15) is therefore a point of $\mathfrak{K}_n$ in every case.

Let π now be a point of the surface of $\mathfrak{K}_n$ with coordinates

$$\gamma_1,\ \gamma_2,\ \dots,\ \gamma_n; \quad \overline{\gamma_1},\ \overline{\gamma}_2,\ \dots,\ \overline{\gamma}_n$$

and

$$\left.\begin{aligned} u_1(c_1-\gamma_1)+u_2(c_2-\gamma_2)+\cdots+u_n(c_n-\gamma_n) \\ +\overline{u}_1(\overline{c}_1-\overline{\gamma}_1)+\overline{u}_2(\overline{c}_2-\overline{\gamma}_2)+\cdots+\overline{u}_n(\overline{c}_n-\overline{\gamma}_n) \end{aligned}\right\} = 0 \tag{16}$$

the equation of a supporting hyperplane through this point. We normalise the coefficients of (16) so that

$$\sum_{k=1}^{n}(u_k^2+\overline{u}_k^2)=1$$

and

$$\sum_{k=1}^{n}(u_k\gamma_k+\overline{u}_k\overline{\gamma}_k)>0.$$

For every point of $\mathfrak{K}_n$ with coordinates $c_k, \overline{c}_k$, the left side of (16) is negative or zero, therefore

$$\sum_{k=1}^{n}(u_k c_k+\overline{u}_k\overline{c}_k)\leqslant\sum_{k=1}^{n}(u_k\gamma_k+\overline{u}_k\overline{\gamma}_k), \tag{17}$$

and for interior points the possibility of equality is ruled out. Because π is now an accumulation point of interior points, then one can determine the quantity

$$\sum_{k=1}^{n}(u_k\gamma_k+\overline{u}_k\overline{\gamma}_k)$$

as the upper limit of

$$\sum_{k=1}^{n}(u_k c_k+\overline{u}_k\overline{c}_k),$$

where the point with coordinates $(c_k, \overline{c}_k)$ runs through the interior of the body $\mathfrak{K}_n$; now, however, every such point is the n-th geometrical representative of a function $y(z)$ which is regular on the unit disc and satsifies our requirements. We can therefore write

$$\sum_{k=1}^{n}(u_k c_k + \overline{u}_k \overline{c}_k) = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\Phi(\theta)\,d\theta \tag{18}$$

by introducing the notation

$$\Phi(\theta) = \sum_{k=1}^{n}(u_k \cos k\theta - \overline{u}_k \sin k\theta), \tag{19}$$

with help from (5). *p.103*

Because now the relations (2) and (4), i.e.

$$\begin{cases} f(\theta) \geqslant 0, \\ \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} f(\theta)\,d\theta = 2, \end{cases}$$

also hold here, the relation

$$\sum_{k=1}^{n}(u_k c_k + \overline{u}_k \overline{c}_k) \leqslant 2M \tag{20}$$

follows from (18), if we denote the maximum of the function $\Phi(\theta)$ on the interval from 0 to 2π by M; the upper limit of the left side of (20) is therefore

$$\sum_{k=1}^{n}(u_k \gamma_k + \overline{u}_k \overline{\gamma}_k) \leqslant 2M. \tag{21}$$

The function $\Phi(\theta)$ is continuous on the interval from 0 to 2π and attains its maximum M for at least one particular value of θ, e.g. for $\theta = \theta_1$, so that

$$M = \sum_{k=1}^{n}(u_k \cos k\theta_1 - \overline{u}_k \sin k\theta_1). \tag{22}$$

On the other hand, the function

$$y = \frac{e_{i\theta_1} + z}{e^{i\theta_1} - z}$$

is regular in the interior of the unit disc, its real part is positive inside this region and it takes the value 1 for $z = 0$. The n-th geometrical representative of this function is therefore a point of $\mathfrak{K}_n$; it has coordinates

$$c_k = 2\cos k\theta_1, \quad \overline{c}_k = -2\sin k\theta_1,$$

and it follows from (22) that for this point

$$\sum_{k=1}^{n}(u_k c_k + \overline{u}_k \overline{c}_k) = 2M.$$

Therefore, due to (17)

$$2M \leqslant \sum_{k=1}^{n}(u_k \gamma_k + \overline{u}_k \overline{\gamma}_k)$$

and then the equality with (21) delivers

$$\sum_{k=1}^{n}(u_k \gamma_k + \overline{u}_k \overline{\gamma}_k) = 2M. \tag{23}$$

p.104 Let

$$P_1,\ P_2,\ P_3,\ \ldots$$

now be a series of infinitely many points in the *interior* of $\mathfrak{K}_n$, which converges to our point π, and let

$$c_1^{(m)},\ c_2^{(m)},\ \ldots,\ c_n^{(m)};\quad \overline{c}_1^{(m)},\ \overline{c}_2^{(m)},\ \ldots,\ \overline{c}_n^{(m)}$$

be the coordinates of the point P_m; due to (17) and (23), we can write

$$\sum_{k=1}^{n}(u_k c_k^{(m)} + \overline{u}_k \overline{c}_k^{(m)}) = 2M - \epsilon_m^2, \tag{24}$$

where ϵ_m denotes a non-zero quantity and

$$\lim_{m\to\infty} \epsilon_m = 0.$$

Furthermore, let y_m be a power series which is convergent on the unit disc, which satisfies all of our requirements and whose geometrical representative is P_m, and $f_m(\theta)$ be the real part of y_m on this disc.

By (18) and (24) the equation

$$\frac{1}{\pi}\int_0^{2\pi} f_m(\theta)\Phi(\theta)\, d\theta = 2M - \epsilon_m^2 \tag{25}$$

holds. Now denote by

$$\theta_1,\ \theta_2,\ \ldots,\ \theta_p$$

all the values of θ between 0 and 2π for which

$$\Phi(\theta) = \sum_{k=1}^{n}(u_k \cos k\theta - \overline{u}_k \sin k\theta)$$

attains its maximal value M; because these quantities consist of an interrupted Fourier series which finishes with the terms in $\cos n\theta$ and $\sin n\theta$, they can never exhibit more than n maxima in the interval from 0 to 2π, which is also true of the values of the constants $u_k, \overline{u}_k$, and so we have

$$p \leqslant n.$$

For sufficiently small values of ϵ_m, or, what amounts to the same, for sufficiently large values of m, the requirement that

$$\Phi(\theta) \geqslant M\epsilon_m$$

defines exactly p distinct disjoint sub-intervals* in the interval from 0 to 2π

$$\delta_1^{(m)},\ \delta_2^{(m)},\ \ldots,\ \delta_p^{(m)};$$

all of these subintervals converge to zero as m tends to infinity. We denote by $L_0^{(m)}$ the value of *p.105*

$$\frac{1}{2\pi}\int f_m(\theta)\,d\theta$$

on the complement of all the subintervals, by $L_j^{(m)}$ the value of the same integral for the interval $\delta_j^{(m)}$. Due to (4), for every m

$$L_0^{(m)} + L_1^{(m)} + \cdots + L_p^{(m)} = 1; \tag{26}$$

on the other hand, we have

$$\frac{1}{\pi}\int_0^{2\pi} f_m(\theta)\Phi(\theta)\,d\theta \leqslant 2L_0^{(m)}(M-\epsilon_m) + 2M(L_1^{(m)} + L_2^{(m)} + \cdots + L_p^{(m)}).$$

From (25) and (26) it therefore follows that

$$2M - \epsilon_m^2 \leqslant 2M - 2L_0^{(m)}\epsilon_m$$

or

$$L_0^{(m)} \leqslant \frac{\epsilon_m}{2};$$

consequently, we also have that

$$\lim_{m\to\infty} L_0^{(m)} = 0. \tag{27}$$

*or at most $p+1$, of which two however contain 0 and 2π.

From the set of functions y_m one can now isolate a subset such that for every

$$j = 1, 2, \ldots, p$$

the $L_j^{(m)}$ converge to particular boundary values; we then have that

$$\lim_{m\to\infty} L_j^{(m)} \lambda_j$$

and, due to (26) and (27),

$$\sum_{j=1}^{p} \lambda_j = 1;$$

the λ_j can, by definition, never be negative, due to the requirement that

$$f_m(\theta) \geqslant 0.$$

We can actually determine the coordinates

$$\gamma_1, \gamma_2, \ldots, \gamma_n; \quad \overline{\gamma}_1, \overline{\gamma}_2, \ldots, \overline{\gamma}_n$$

of our boundary point π using the results we have found, because

$$\gamma_k = \lim_{m\to\infty} c_k^{(m)}, \quad \overline{\gamma}_k = \lim_{m\to\infty} \overline{c}_k^{(m)};$$

p.106

$$c_k^{(m)} = \frac{1}{\pi} \int_0^{2\pi} f_m(\theta) \cos k\theta \, d\theta,$$
$$\overline{c}_k^{(m)} = \frac{-1}{\pi} \int_0^{2\pi} f_m(\theta) \sin k\theta \, d\theta$$

and, if we allow the intervals $\delta_1^{(m)}, \delta_2^{(m)}, \ldots, \delta_p^{(m)}$ to become infinitesimally small as m increases, the middle statement gives

$$\begin{cases} \gamma_k = 2 \sum_{j=1}^{p} \lambda_j \cos k\theta_j, \\ \overline{\gamma}_k = -2 \sum_{j=1}^{p} \lambda_j \sin k\theta_j. \end{cases}$$

If we now remark that p is at most equal to n, then we see that *every* point π of the surface of $\mathfrak{D}_n$ of $\mathfrak{K}_n$ has coordinates

$$\text{(28)} \qquad \begin{cases} \gamma_k = 2 \sum_{j=1}^{n} \lambda_j \cos k\theta_j, \\ \overline{\gamma}_k = -2 \sum_{j=1}^{n} \lambda_j \sin k\theta_j, \end{cases}$$

where θ_j, λ_j denote appropriate constants which satisfy the conditions

(29) $$\begin{cases} 0 \leqslant \theta_j \leqslant 2\pi, \quad \lambda_j \geqslant 0 \quad j = 1, 2, \dots, n \\ \sum_{j=1}^{n} \lambda_j = 1. \end{cases}$$

If, conversely, θ_j and λ_j are $2n$ arbitrary real numbers which meet the conditions (29), then the equations (28) represent the coordinates of a point P of $\mathfrak{K}_n$, because P is the n-th geometrical representative of the function

(30) $$y = \sum_{j=1}^{n} \lambda_j \frac{e^{i\theta_j} + z}{e^{i\theta_j} - z},$$

and this satisfies all of our conditions: it is regular and its real part is positive for

$$|z| < 1,$$

and for $z = 0$ we have $y = 1$. *p.107*

Therefore, if one allows the values

$$\theta_1, \theta_2, \dots, \theta_n,$$
$$\lambda_1, \lambda_2, \dots, \lambda_n$$

vary unconstrained within the limits of (29), then the point whose coordinates are given by the equations

(31) $$\begin{cases} \gamma_k = 2 \sum_{j=1}^{n} \lambda_j \cos k\theta_j, \\ \overline{\gamma}_k = -2 \sum_{j=1}^{n} \lambda_j \sin k\theta_j \end{cases}$$

never leaves the convex body $\mathfrak{K}_n$ and every point of the surface $\mathfrak{D}_n$ of this body is attained by at least one system of values θ_j, λ_j.

§4.
Uniqueness.

It must now be shown that the rational functions of the form (30) are the *only* ones which satisfy all our conditions and have a point π of the surface $\mathfrak{D}_n$ of $\mathfrak{K}_n$ as a geometrical representative. We assume that $y(z)$ is a function which achieves this; then the function $y = y(rz)$ is also regular on the unit disc for every $r < 1$

and we can introduce the notation $f(r,\theta)$ for the real part of this last function on the unit disc. We shall immediately make use of the observation that $f(r,\theta)$ also represents the real part of the original function $y(z)$ on the circle $re^{i\theta}$.

Because the representative of $y(rz)$ lies in the *interior* of $\mathfrak{K}_n$, we have the analogous equation to equation (25),

$$\frac{1}{\pi}\int_0^{2\pi} f(r,\theta)\Phi(\theta)\,d\theta = M - \epsilon^2(r),$$

where $\epsilon(r)$ is a non-zero positive value and, because the n-th geometrical representative of $y(rz)$ changes continuously with r,

$$\lim_{r=1} \epsilon(r) = 0.$$

We again consider the intervals

$$\delta_1(r),\ \delta_2(r),\ \ldots,\ \delta_p(r),$$

inside which

$$\Phi(\theta) \geqslant M - \epsilon(r),$$

p.108 and denote by $L_j(r)$ the value of the integral

$$\int f(r,\theta)\,d\theta$$

on the interval $\delta_j(r)$. One can always specify a series of increasing positive quantities

$$r_1,\ r_2,\ r_3,\ \ldots \tag{32}$$

which are all < 1, converge to one and have the property that for every

$$j = 1, 2, \ldots, p,$$

the limit

$$\lim_{n=\infty} L_j(r_n) = \lambda_j$$

exists.

We now consider the function

$$\overline{y}(z) = \sum_{j=1}^{p} \lambda_j \frac{e^{i\theta_j} + z}{e^{i\theta_j} - z}$$

where the λ_j are given by the values specified and the θ_j have the same meaning as earlier; the quantities λ_j, θ_j satisfy the relation (29) by construction, so that

$\overline{y}(z)$ satisfy all our conditions. We denote the real part of $\overline{y}(z)$ on the disc $re^{i\theta}$ by $\overline{f}(r,\theta)$.

If

$$z = \rho e^{i\psi}$$

is now an arbitrary point in the interior of the disc $|z| < 1$, then in the series (32) all $r_n > \rho$ for sufficiently large n, and $f(\rho,\psi)$ can be expressed by the Poisson integral for a disc with radius r_n

$$f(\rho,\psi) = \frac{1}{2\pi}\int_0^{2\pi} \frac{f(r_n,\theta)(r_n^2-\rho^2)}{r_n^2 - 2r_n\rho\cos(\theta-\psi)+\rho^2}\, d\theta.$$

But now the left hand side of this equation is independent of n and we obtain

$$f(\rho,\psi) = \sum_{j=1}^{p} \frac{\lambda_j(1-\rho^2)}{1-2\rho\cos(\theta_j-\psi)+\rho^2},$$

if we let n tend to infinity and approximate the integral similarly to earlier. We would obtain exactly the same expression from the calculation of $\overline{f}(\rho,\psi)$, so that consequently the equation

$$f(\rho,\psi) = \overline{f}(\rho,\psi)$$

must be an identity in ρ and ψ; from here it finally follows, because *p.109*

$$y(0) = \overline{y}(0) = 1,$$

that

$$y(z) = \sum_{j=1}^{p} \lambda_j \frac{e^{i\theta_j}+z}{e^{i\theta_j}-z},$$

i.e. the relation that we wanted to prove.

At the same time we are given that the points of the surface $\mathfrak{D}_n$ may be represented in a unique way by the formula (28); indeed, for each way in which the quantities (λ_j,θ_j) and (λ'_j,θ'_j) may be substituted into the formulas, the same values for $\gamma_k, \overline{\gamma}_k$ are given, and, by the foregoing, the equation

$$\sum_{j=1}^{n} \lambda_j \frac{e^{i\theta_j+z}}{e^{i\theta_j}-z} = \sum_{j=1}^{n} \lambda'_j \frac{e^{i\theta_j+z}}{e^{i\theta_j}-z}$$

must be satisfied by z, from which it follows that the λ_j,θ_j and λ'_j,θ'_j are identical up to ordering.

We therefore have the theorem: *There is only one function whose representative is a given point of the surface $\mathfrak{D}_n$ of $\mathfrak{K}_n$, and this is rational and of degree at most n.*

§5.
Completeness of the representation.

In order to also prove the converse of this theorem, i.e. that the representative of every rational function which can be represented in the form (30) belongs to the surface $\mathfrak{D}_n$, we must consider the geometrical properties of $\mathfrak{K}_n$ somewhat more closely.

We first remark that every point of the surface or of the interior of the convex body $\mathfrak{K}_n$ can be viewed as the projection to $2n$-dimensional space of a point of the surface $\mathfrak{D}_{n+1}$ (which lies in $(2n+2)$-dimensional space). From here it follows that the coordinates of every point of $\mathfrak{K}_n$ can be represented in the form

$$\text{(33)} \qquad \begin{cases} c_k = 2 \sum_{j=1}^{(n+1)} \lambda_j \cos k\theta_j, \\ \bar{c}_k = -2 \sum_{j=1}^{(n+1)} \lambda_j \sin k\theta_j, \\ \qquad k = 1, 2, \ldots, n, \end{cases}$$

p.110 where the λ_j are all either positive or zero and their sum is equal to one.

We denote by $\mathfrak{C}_n$ the curve which is described by the points with coordinates

$$\begin{array}{c} 2\cos\theta, \quad 2\cos 2\theta, \; \ldots, \quad 2\cos n\theta, \\ -2\sin\theta, -2\sin 2\theta, \; \ldots, -2\sin n\theta \end{array}$$

in $2n$-dimensional space, if θ varies from 0 to 2π. The points of this curve all belong to $\mathfrak{K}_n$ as representatives of the functions

$$y(z) = \frac{e^{i\theta} + z}{e^{i\theta} - z}.$$

If

$$C_1, \, C_2, \, \ldots, \, C_p$$

are the points of $\mathfrak{C}_n$ which correspond to the values

$$\theta_1, \, \theta_2, \, \ldots, \, \theta_p$$

of θ which occur in (33), (where it is assumed that

$$p \leqslant n + 1$$

and

$$\lambda_{p+1} = \lambda_{p+2} = \cdots = \lambda_{n+1} = 0),$$

then the point with coordinates (33) also belongs to *every* convex body to which the points $C_1, C_2, \ldots, C_p$ belong; this can be proven without anything further with the help of our definition of a convex body and the application of the conclusion for n to $(n+1)$.

From all these facts it follows that *every* point of $\mathfrak{K}_n$ is contained in *every* convex body which contains the curve $\mathfrak{C}_n$, that is, *that $\mathfrak{K}_n$ is the smallest convex body which contains $\mathfrak{C}_n$.*

Every supporting hyperplane of $\mathfrak{C}_n$ is therefore a supporting hyperplane of $\mathfrak{K}_n$, else the part $\mathfrak{K}'_n$ of $\mathfrak{K}_n$ which lay on the same side of the supporting hyperplanes as $\mathfrak{C}_n$ would already form a convex body which contained $\mathfrak{C}_n$ and which was smaller than $\mathfrak{K}_n$. Similarly, we see that every supporting hyperplane of $\mathfrak{K}_n$ must also be a supporting hyperplane of $\mathfrak{C}_n$.

Now let P be a point with coordinates given as

$$(34)\qquad \begin{cases} \gamma_k = 2\sum\limits_{j=1}^{p} \lambda_j \cos k\theta_j, \quad \overline{\gamma}_k = -2\sum\limits_{j=1}^{p} \lambda_j \sin k\theta_j \\ \qquad\qquad k = 1, 2, \ldots, n \\ \qquad\qquad p \leqslant n, \end{cases}$$

where the λ_j are always positive and have one as their sum.

I shall show that this point is a point of the surface $\mathfrak{D}_n$ of $\mathfrak{K}_n$; this can be proven by constructing a supporting hyperplane of $\mathfrak{K}_n$, or, equivalently, of $\mathfrak{C}_n$, which contains the point P. *p.111*

For this purpose we consider a series of n distinct points of $\mathfrak{C}_n$, which encompass the points corresponding to the values

$$\theta_1,\ \theta_2,\ \ldots,\ \theta_p$$

in (34).

Every $(2n-1)$-dimensional hyperplane which contains these n points must go through P. Now, however, there is one such hyperplane with the equation

$$(35)\qquad \sum_{k=1}^{n} (u_k c_k + \overline{u}_k \overline{c}_k) = 2d,$$

which touches the curve $\mathfrak{C}_n$ at each of the n points; one only needs the $2n$ linear equations

$$\begin{cases} \sum\limits_{k=1}^{n} (u_k \cos k\theta_j - \overline{u}_k \sin k\theta_j) = d, \\ \sum\limits_{k=1}^{n} (k u_k \sin k\theta_j + k\overline{u}_k \cos k\theta_j) = 0, \\ \qquad\qquad j = 1, 2, \ldots, n \end{cases}$$

to determine the ratios of the $2n+1$ values $u_k, \overline{u}_k$ and d.

The expression

$$\sum_{k=1}^{n} (u_k \cos k\theta - \overline{u}_k \sin k\theta) = d \tag{36}$$

has at most $2n$ real roots, as θ varies from 0 to 2π; it can therefore only vanish in this interval for the values

$$\theta_1,\, \theta_2,\, \ldots,\, \theta_n,$$

because each of these values corresponds to a double root. It has the same sign in the neighbourhood of each of these values. In other words, (35) represents the desired supporting hyperplane of $\mathfrak{C}_n$ which contains the point P.

From here it now follows that the equations (28) *consequently* can represent points of the surface $\mathfrak{D}_n$, and because we proved the uniqueness of the representation for these points in §4, it follows in complete generality that equations such as

$$\begin{gathered}\sum_{j=1}^{n} (\lambda_j \cos k\theta_j - \mu_j \cos k\psi_j) = 0,\\ \sum_{j=1}^{n} (\lambda_j \sin k\theta_j - \mu_j \sin k\psi_j) = 0,\\ k = 1, 2, \ldots, n\end{gathered}$$

p.112 with the additional conditions

$$\begin{cases} 0 \leqslant \theta_j < 2\pi, \quad 0 \leqslant \psi_j < 2\pi,\\ \qquad \lambda_j \geqslant 0, \quad \mu_j \geqslant 0,\\ \sum_{j=1}^{n} \lambda_j = 1, \quad \sum_{j=1}^{n} \mu_j = 1\end{cases}$$

can only hold if, with a suitable ordering of μ_j, ψ_j,

$$\begin{cases} \lambda_j = \mu_j, \qquad \theta_j = \psi_j\\ \quad j = 1, 2, \ldots, n.\end{cases}$$

Therefore, if one varies the θ_j, λ_j in the formula (28) subject to the conditions (29), then the surface $\mathfrak{D}_n$ is described as the region of variability of the coefficients, and the relation between the points of this surface and the system of values for the λ_j, θ_j is a bijective one.

Every point of the region of variability $\mathfrak{K}_n$ is further attained if we replace the condition

$$\sum_{j=1}^{n} \lambda_j = 1$$

by the other

$$\sum_{j=1}^{n} \lambda_j \leqslant 1.$$

We therefore have the following theorem:

Every point P of the region of variability $\mathfrak{K}_n$ is the n-th geometrical representative of one and only one rational function of the form

$$(37) \qquad \begin{cases} y = \lambda_0 + \sum_{j=1}^{n} \lambda_j \dfrac{e^{i\theta_j} + z}{e^{i\theta_j} - z}, \\ \lambda_j \geqslant 0, \quad j = 0, 1, 2, \ldots, n \\ \qquad \sum_{j=0}^{n} \lambda_j = 1; \end{cases}$$

for $\lambda_0 = 0$ the point P lies on the surface $\mathfrak{D}_n$ of $\mathfrak{K}_n$ and the functions (37) are the unique ones whose representative lies on $\mathfrak{D}_n$.

§6.
General problems.

p.113

We now want to apply the results discovered to general problems and assume that T is a region of a hyperplane of a complex variable u, or also a Riemann surface which lies in this hyperplane, and we can map the halfplane

$$\mathfrak{R}(y) \geqslant 0$$

to this region with the help of an analytic function $\phi(y)$; thereby the point $y = 1$ transforms into a regular point

$$u = M_0 = m_0 + i\overline{m}_0$$

of the interior of T. The function $\phi(y)$ can be expanded into a convergent power series of the form

$$(38) \qquad u = M_0 + \sum_{k=1}^{\infty} M_k (y-1)^k$$

in the neighbourhood of the point $y = 1$, where

$$|M_1| \neq 0;$$

one can therefore invert the series (38) in the neighbourhood of $u = M_0$ and e.g. write

$$y - 1 = \sum_{k=1}^{\infty} N_k (u - M_0)^k \tag{39}$$

or

$$y = \psi(u).$$

Now let $u(z)$ be an analytic function of the complex variable z which for all values

$$|z| < 1$$

remains bounded in the interior of the region T, is regular in this region, and takes the value $u = M_0$ for $z = 0$; by the assumptions made this function can be expanded into the convergent power series

$$u = M_0 + \sum_{k=1}^{\infty} A_k z^k \tag{40}$$

in the neighbourhood of $z = 0$, and we want to investigate the region of variability of the n-th geometrical representative of this power series.

If we substitute into (39) a given function $u(z)$ for u, then y becomes a function p.114 of z which satisfies all the requirements described at the beginning of this work. We can therefore expand it into the convergent power series

$$y = 1 + \sum_{k=1}^{\infty} C_k z^k \tag{41}$$

and the n-th geometrical representative of this power series will lie in the interior or on the surface of our earlier body $\mathfrak{K}_n$.

The equations (38) and (39), joined with (40) and (41), now give

$$\sum_{k=1}^{\infty} A_k z^k = \sum_{j=1}^{\infty} M_j \left[\sum_{l=1}^{\infty} C_l z^l \right]^j,$$

$$\sum_{k=1}^{\infty} C_k z^k = \sum_{j=1}^{\infty} N_j \left[\sum_{l=1}^{\infty} A_l z^l \right]^j,$$

and we can, with the help of these last relations, if one compares the same powers of z on the right hand side and the left hand side, also express A_k as a completely rational function of $C_1, C_2, \ldots, C_k$, as well as express C_k as a completely rational function of $A_1, A_2, \ldots, A_k$.

The n-th geometrical representatives of the power series (40) and (41) therefore correspond to each other in a bijective way and the birational transformation of $2n$-dimensional space which realises this map only depends on all real and imaginary parts of the first n coefficients $M_1, M_2, \ldots, M_n$ of the power series (38). We now obtain the domain of variability K_n of the n coefficients $A_1, A_2, \ldots, A_n$ by applying the transformation to our earlier body $\mathfrak{K}_n$ and consequently the surface Ω_n of K_n contains the representatives of functions of the form

$$\phi\begin{pmatrix}\alpha_0+\alpha_1 z+\cdots+\alpha_n z^n\\ \beta_0+\beta_1 z+\cdots+\beta_n z^n\end{pmatrix}, \tag{42}$$

where $\phi(y)$ again denotes a function which maps the half plane to the region T, and α, β denote constants.

All our earlier theorems may be transferred to the body K_n in a suitably modified form. In particular the coordinates of the surface Ω_n of K_n may be expressed as rational functions of the parameters

$$\begin{cases}\lambda_j, \quad \cos k\theta_j, \quad \sin k\theta_j,\\ \qquad j = 1, 2, \ldots, n\\ \qquad k = 1, 2, \ldots, n\end{cases}$$

used earlier. Thereby, by the fact that the transformation is birational, we ensure *p.115* that two points of Ω_n which correspond to different values of the parameters can never coincide, and that consequently the surface Ω_n does not intersect itself and the body K_n is simply connected.

As I suggested in my Comptes Rendus Note (26 December 1905), one can also extend the theory further to such cases in which the point $u = M_0$ of the region T is singular, but the function $u(r)$ remains solely in T; then the function $u(z)$ in fact possesses a singularity at $z = 0$, but the power series (41) is nevertheless regular at this point and this suffices to establish the body K_n here as well.

Brussels, 14th September 1906.